3.2 Properties of Determinants

Key idea: If two matrices are row equivalent, then their determinants are related in precise ways. We mention these here and use them to compute determinants much more efficiently.

Recall that $\det(A)$ was very easy to compute in the case that A was triangular. Thus, if we knew how the determinant of a matrix was related to the determinant of an echelon form, the latter would be simple to compute making the former simple as well.
Indeed, we do know how these relate.

Write before lecture if possible

Fact: (The determinant and row reduction).
For A a square matrix

1) If a multiple of a row of A is added to another to produce a matrix B, then $\det(A) = \det(B)$.

2) If two rows of A are swapped to produce B then $\det(B) = -\det(A)$.

3) If one row of A is multiplied by k to produce B then $\det(B) = k \cdot \det(A)$

These properties make sense if you reconsider the definition of $\det(A)$ in light of them: e.g. prop 2 follows from rearranging where the cofactors are

$$(a_{1j} \leftrightarrow a_{2j} \Rightarrow C_{1j} \leftrightarrow -C_{2j}).$$

Let's now use these properties to compute some determinants:

Method: we reduce a matrix to echelon form, tracking how each operation affects the determinant. The matrix whose determinant we actually compute is triangular (so it's the product of the diagonal entries).

Ex) Compute the determinant of

$$A=\begin{bmatrix}1&-4&2\\-2&8&-9\\-1&7&0\end{bmatrix}.$$

We "track" the reduction of A to make this an easier calculation

$$\det(A)=\begin{vmatrix}1&-4&2\\-2&8&-9\\-1&7&0\end{vmatrix}\underset{\substack{\text{combining}\\\text{rows doesn't}\\\text{affect the}\\\text{determinant}}}{=}\begin{vmatrix}1&-4&2\\0&0&-5\\0&3&2\end{vmatrix}\underset{\substack{\text{to get}\\\text{echelon}\\\text{form, we}\\\text{swap R2}\\\text{and R3}}}{=}(-1)\begin{vmatrix}1&-4&2\\0&3&2\\0&0&-5\end{vmatrix}$$

$$=(-1)(1)(3)(-5)=\boxed{15}$$

det. of triangular matrices are simple!

Ex) Compute $\det(A)$ if

$$A=\begin{bmatrix}2&-8&6&8\\3&-9&5&10\\-3&0&1&-2\\1&-4&0&6\end{bmatrix}.$$

$$\det(A)=\begin{vmatrix}2&-8&6&8\\3&-9&5&10\\-3&0&1&-2\\1&-4&0&6\end{vmatrix}\underset{\substack{\text{swap R1, R4}\\\text{-or-}\\\text{Factor out}\\2}}{=}2\begin{vmatrix}1&-4&3&4\\3&-9&5&10\\-3&0&1&-2\\1&-4&0&6\end{vmatrix}\underset{\substack{\text{combining}\\\text{rows}}}{=}2\begin{vmatrix}1&-4&3&4\\0&3&-4&-2\\0&-12&10&10\\0&0&-3&2\end{vmatrix}$$

Mult. by $\frac{1}{2}$ => $\frac{1}{2}\det A=\det B$
so $\det A = 2\det B$

$$=2\begin{vmatrix}1&-4&3&4\\0&3&-4&-2\\0&0&-6&2\\0&0&-3&2\end{vmatrix}=2\begin{vmatrix}1&-4&3&4\\0&3&-4&-2\\0&0&-6&2\\0&0&0&1\end{vmatrix}$$

$$=2(1)(3)(-6)(1)=\boxed{-36}$$

Thus we see in general that the determinant of A is linked closely to it's echelon form:

Fact: If A is reduced to U without scaling any rows and U is in echelon form then

(note: this is always possible.)

$$\det(A)=\begin{cases}(-1)^r\cdot\begin{pmatrix}\text{product of}\\\text{pivots in } U\end{pmatrix} & \text{if } A \text{ invertible}\\ 0 & \text{if } A \text{ noninvertible}\end{cases}$$

A invertible $U=\begin{bmatrix}p&*&*\\0&p&*\\0&0&p\end{bmatrix}$

A noninvertible $U=\begin{bmatrix}p&*&*\\0&p&*\\0&0&0\end{bmatrix}$

On a numerical note: recall an $n \times n$ determinant takes $n!$ operations to compute via the definition. The method described here requires much less at $\frac{2}{3}n^3$ many operations. Thus, we need 10,000 operations for a 25×25 matrix compared to 1.5×10^{25} many. (less than a second vs 500,000 years.)

This method also leads to an important characterization of invertibility:

Fact: A is invertible if and only if $\det(A) \neq 0$.

Recall section 2.2 and 2.3 matrix inverses.

(So we gain an additional item in the invertible matrix theorem.)

To conclude, two final properties of the determinant:

Fact: 1) If A is $n \times n$, $\det(A^T) = \det(A)$

so all the facts about row operations hold about column operations
e.g. $\det[a_1 \; a_2] = -\det[a_2 \; a_1]$.

2) If A, B are $n \times n$, $\det(AB) = \det(A)\det(B)$.

Ex: $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ $\Rightarrow$ $AB = \begin{bmatrix} 2 & 7 \\ 8 & 8 \end{bmatrix}$ $\rightsquigarrow$ $\det(AB) = 16 - 56 = -40$

$\det(A)\det(B) = -10 \cdot 4 = -40$

$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$\det(A) = 2 - 12 = -10$ $\quad$ $\det(B) = 4 - 0 = 4$

NOTE: $\det(A+B) \neq \det(A) + \det(B)$ in general.

If you're curious, the determinant defines a (multi) linear transformation for every matrix A. This fact is studied in deeper analyses of the determinant. (One can show in some sense that the determinant is the only function that can do this.)